EMORY UNIVERSITY SCHOOL OF MEDICINE

Department of Biomedical Informatics

Kalman Filtering in Biomedical Applications

From theory to practice

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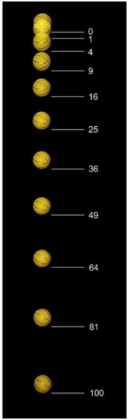
Outline

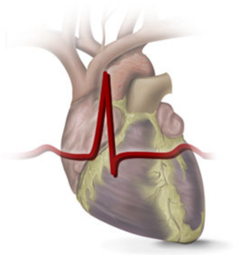
1. Kalman filtering basics
2. Case studies
   - Powerline cancellation
   - Electroencephalogram analysis
   - Adult and fetal electrocardiogram processing
   - Phonocardiogram processing
   - Pandemic trend forecasting
3. Advanced Kalman filtering
   - The Kalman filter revisited
   - Kalman filter engineering
4. Introduction to the lab session

Outline

Dynamics

Dynamics: Evolution of a signal/system in time. Any movement/change comprises a dynamics.





Dynamic equations

Dynamic equations: Differential/difference equations used to formulate a dynamic system , e.g.

Constant acceleration motion (linear-continuous)

$$a\ddot{x}(t) + b\dot{x}(t) + cx(t) = 0$$

The Van der Pol model for action potentials (nonlinear-continuous)

$$\ddot{x}(t) - 2\alpha[1 - x(t)^2]\dot{x}(t) + \omega_0^2 x(t) = 0$$

First-order lowpass filter or an autoregressive model (linear-discrete)

$$x_{n+1} = \alpha x_n + u_n$$

The Logistic Map in biological growth (nonlinear-discrete)

$$x_{n+1} = r x_n(1 - x_n)$$

State variables

State

- **States** are properties of a system (usually represented as a vector), through which the system's response to a given input can be uniquely found.
- Knowledge of a system's dynamics and its states at time t_0, makes us needless of its past $(t < t_0)$, i.e. it is not important how the system reached that state.
- In physical/biological systems, storage of the 'past information' requires energy saving elements (capacitors, inductors, neurons, cells, etc.). In digital systems memory elements (registers) are required to preserve states.

State variables

(continued)

Example: First-order system

$$\dot{x}(t) + \alpha x(t) = \beta u(t)$$

This system has one state; knowledge of $x(t_0)$ is sufficient to determine $x(t)$ for all $t \in [t_0, \infty)$.

Example: continuous-time delay

$$x(t) = u(t - t_0)$$

This system has infinite number of states, but can be approximated with low-order state-space models when the input is *low frequency* ($|f_{\max} t_0|$ is small).

State-space representations

Dynamic systems (difference or differential) can be represented in state-space form.

Continuous-time state-space equations

A set of first-order differential (difference) equations relating the system inputs, outputs, and states

$$\begin{aligned} \dot{\boldsymbol{s}}(t) &= \boldsymbol{f}(\boldsymbol{s}(t), \boldsymbol{u}(t)) \qquad \textbf{state equation} \\ \boldsymbol{x}(t) &= \boldsymbol{g}(\boldsymbol{s}(t), \boldsymbol{u}(t)) \qquad \textbf{observation equation} \end{aligned}$$

where $\boldsymbol{u}(t)$, $\boldsymbol{x}(t)$, and $\boldsymbol{s}(t)$ are the input, output and state vectors.

Discrete-time state-space equations

$$\begin{aligned} \boldsymbol{s}_{n+1} &= \boldsymbol{f}(\boldsymbol{s}_n, \boldsymbol{u}_n) \qquad \textbf{state equation} \\ \boldsymbol{x}_n &= \boldsymbol{g}(\boldsymbol{s}_n, \boldsymbol{u}_n) \qquad \textbf{observation equation} \end{aligned}$$

- State-space representations are not unique; but some *canonical forms* are more common
- For linear time-invariant (LTI) systems, differential/difference and state-space equations can be converted to one another.

Stochastic models and state-space representations

Stochastic dynamic systems

Dynamic systems can be driven by stochastic inputs or observed through noisy measurements.

- Continuous case:

$$\begin{aligned}\dot{\boldsymbol{s}}(t) &= \boldsymbol{f}(\boldsymbol{s}(t), \boldsymbol{u}(t), \boldsymbol{w}(t))\\ \boldsymbol{x}(t) &= \boldsymbol{g}(\boldsymbol{s}(t), \boldsymbol{u}(t), \boldsymbol{v}(t))\end{aligned}$$

- Discrete case:

$$\begin{aligned}\boldsymbol{s}_{n+1} &= \boldsymbol{f}(\boldsymbol{s}_n, \boldsymbol{u}_n, \boldsymbol{w}_n)\\ \boldsymbol{x}_n &= \boldsymbol{g}(\boldsymbol{s}_n, \boldsymbol{u}_n, \boldsymbol{v}_n)\end{aligned}$$

$\boldsymbol{w}(t)$ and $\boldsymbol{w}_n$ are **process noise**; $\boldsymbol{v}(t)$ and $\boldsymbol{v}_n$ are **observation (measurement) noise**.

State estimation

Solving dynamic systems

- Deterministic dynamic (or state-space) systems can be solved by having their initial conditions.
- Stochastic dynamic systems can only be **estimated**, under some measure of *optimality*

Examples

- The **Kalman filter** provides the *minimum mean square error* (MMSE) estimate of the state vector
- The **H-infinity (H_∞) filter** gives the *minimum maximum error* estimate (minimizes the worst case estimation error, instead of the variance of the error)

The classical linear Kalman filter

For the discrete-time stochastic dynamic system:

$$\boldsymbol{s}_{n+1} = \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{B}_n \boldsymbol{w}_n$$
$$\boldsymbol{x}_n = \boldsymbol{C}_n \boldsymbol{s}_n + \boldsymbol{D}_n \boldsymbol{v}_n$$

where $\boldsymbol{w}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_n)$, $\boldsymbol{v}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}_n)$ and $\hat{\boldsymbol{s}}_0 = \mathbb{E}\{\boldsymbol{s}_0\}$, the Kalman filter is a set of recursive equations for estimating the state vector $\boldsymbol{s}_n$:

1. **Time propagation:**

$$\hat{\boldsymbol{s}}^-_{n+1} = \boldsymbol{A}_n \hat{\boldsymbol{s}}_n$$
$$\boldsymbol{P}^-_{n+1} = \boldsymbol{A}_n \boldsymbol{P}_n \boldsymbol{A}_n^T + \boldsymbol{B}_n \boldsymbol{Q}_n \boldsymbol{B}_n^T$$

2. **Kalman gain:** $\boldsymbol{K}_n = \boldsymbol{P}^-_n \boldsymbol{C}_n^T (\boldsymbol{C}_n \boldsymbol{P}^-_n \boldsymbol{C}_n^T + \boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T)^{-1}$

3. **Measurement propagation:**

$$\hat{\boldsymbol{s}}_n = \hat{\boldsymbol{s}}^-_n + \boldsymbol{K}_n(\boldsymbol{x}_n - \boldsymbol{C}_n \hat{\boldsymbol{s}}^-_n)$$
$$\boldsymbol{P}_n = \boldsymbol{P}^-_n - \boldsymbol{K}_n \boldsymbol{C}_n \boldsymbol{P}^-_n$$

Rudolf Emil Kálmán
May 19, 1930 (Budapest, Hungary)–July 2, 2016 (Gainesville, Florida)
Electrical Engineer and Mathematician.
Has significant contributions in Applied Engineering Systems Theory. Is best known for the Kalman filter: Kalman, R.E. (1960). "A New Approach to Linear Filtering and Prediction Problems". Journal of Basic Engineering. 82 (1): 35–45. `doi:10.1115/1.3662552`

The classical linear Kalman filter
(continued)

The intuition behind the Kalman filter

- The Kalman filter is a **recursive Bayesian estimator**.
- It estimates states as an "optimal" **weighted combination** of **priors** (knowledge obtained from the system's dynamic model) and **posteriors** (through noisy measurements of the states).
- When the measurements are "clean" trust the observations (posteriors); when they are "noisier" rely on the model (priors).
- When the model is "accurate" trust the priors; when the model is imprecise tend towards the observations.

Nonlinear extensions of the Kalman filter

- The Kalman filter has several extensions for nonlinear systems. A very practical one is the **extended Kalman filter (EKF)**, with extensive applications in biomedical systems (*since biological systems are never linear*)

linearization

$$\begin{aligned} \boldsymbol{s}_{n+1} &= \boldsymbol{f}_n(\boldsymbol{s}_n, \boldsymbol{w}_n) \\ \boldsymbol{x}_n &= \boldsymbol{g}_n(\boldsymbol{s}_n, \boldsymbol{v}_n) \end{aligned}$$

$$\begin{aligned} \boldsymbol{s}_{n+1} &\approx \boldsymbol{f}_n(\hat{\boldsymbol{s}}_n, \bar{\boldsymbol{w}}_n) + \boldsymbol{A}_n(\boldsymbol{s}_n - \hat{\boldsymbol{s}}_n) + \boldsymbol{B}_n(\boldsymbol{w}_n - \bar{\boldsymbol{w}}_n) \\ \boldsymbol{x}_n &\approx \boldsymbol{g}_n(\hat{\boldsymbol{s}}_n, \bar{\boldsymbol{v}}_n) + \boldsymbol{C}_n(\boldsymbol{s}_n - \hat{\boldsymbol{s}}_n) + \boldsymbol{D}_n(\boldsymbol{v}_n - \bar{\boldsymbol{v}}_n) \end{aligned}$$

$$\boldsymbol{A}_n = \left.\frac{\partial \boldsymbol{f}_n(\boldsymbol{s}, \bar{\boldsymbol{w}}_n)}{\partial \boldsymbol{s}}\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_n} \qquad \boldsymbol{B}_n = \left.\frac{\partial \boldsymbol{f}_n(\hat{\boldsymbol{s}}_n, \boldsymbol{w})}{\partial \boldsymbol{w}}\right|_{\boldsymbol{w}=\bar{\boldsymbol{w}}_n}$$

$$\boldsymbol{C}_n = \left.\frac{\partial \boldsymbol{g}_n(\boldsymbol{x}, \bar{\boldsymbol{v}}_n, n)}{\partial \boldsymbol{s}}\right|_{\boldsymbol{s}=\hat{\boldsymbol{s}}_n} \qquad \boldsymbol{D}_n = \left.\frac{\partial \boldsymbol{g}_n(\hat{\boldsymbol{s}}_n, \boldsymbol{v})}{\partial \boldsymbol{v}}\right|_{\boldsymbol{v}=\bar{\boldsymbol{v}}_n}$$

The extended Kalman filter (EKF)

$$\begin{aligned} \hat{\boldsymbol{s}}_{n+1}^- &= \boldsymbol{f}_n(\hat{\boldsymbol{s}}_n, \bar{\boldsymbol{w}}) \\ \boldsymbol{i}_n &= \boldsymbol{x}_n - \boldsymbol{g}_n(\hat{\boldsymbol{s}}_n^-, \bar{\boldsymbol{v}}) \\ \boldsymbol{K}_n &= \boldsymbol{P}_n^- \boldsymbol{C}_n^T (\boldsymbol{C}_n \boldsymbol{P}_n^- \boldsymbol{C}_n^T + \boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T)^{-1} \\ \hat{\boldsymbol{s}}_n^+ &= \hat{\boldsymbol{s}}_n^- + \boldsymbol{K}_n \boldsymbol{i}_n \\ \boldsymbol{P}_{n+1}^- &= \boldsymbol{A}_n \boldsymbol{P}_n^+ \boldsymbol{A}_n^T + \boldsymbol{B}_n \boldsymbol{Q}_n \boldsymbol{B}_n^T \\ \boldsymbol{P}_n^+ &= \boldsymbol{P}_n^- - \boldsymbol{K}_n \boldsymbol{C}_n \boldsymbol{P}_n^- \end{aligned}$$

Outline

Kalman filters for biomedical applications

- While the Kalman filter (KF) is a classical tool, it can be adapted to new applications with different dynamics.
- Dynamic models for biological systems are not unique. Therefore, numerous KFs can be developed for a single application.
- Hence, besides the engineering aspects of a well-functioning KF, the performance highly replies on the *knowledge* and *art* of **biological system modeling**.
- In the sequel, we study various case studies of KFs in biomedical applications.

Outline

Kalman filters for powerline noise estimation and removal from biosignals

We study a Kalman filter for powerline noise estimation and removal from biosignals [Sameni, 2012].

Data model

- Biosignal v_n contaminated by powerline x_n:
$$y_n = x_n + v_n$$
- Powerline signal model:
$$x_n = B\cos(\omega_0 n + \phi)$$
- $\omega_0 = 2\pi f_0 / f_s$
- f_0 powerline frequency
- f_s the sampling frequency
- B amplitude of the contaminated powerline over the biosignal v_n

We can show that:

$$x_{n+1} + x_{n-1} = 2\cos(\omega_0)x_n$$

Kalman filters for powerline noise estimation and removal from biosignals (continued)

State-space form of the data model

$$\begin{aligned} \boldsymbol{x}_{n+1} &= \boldsymbol{A}\boldsymbol{x}_n + \boldsymbol{b}w_n \\ y_n &= \boldsymbol{h}^T\boldsymbol{x}_n + v_n \end{aligned}$$

where $\boldsymbol{x}_n = [x_n \quad x_{n-1}]^T$, $\boldsymbol{A} = \begin{bmatrix} 2\cos(\omega_0) & -1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{b} = [1 \quad 0]^T$, and $\boldsymbol{h} = [1 \quad 0]^T$.

Kalman filter equations

$$\begin{aligned} \hat{\boldsymbol{x}}_{n+1}^- &= \boldsymbol{A}\hat{\boldsymbol{x}}_n \\ \boldsymbol{P}_{n+1}^- &= \boldsymbol{A}\boldsymbol{P}_n\boldsymbol{A}^T + q_n\boldsymbol{b}\boldsymbol{b}^T \end{aligned} \qquad \boldsymbol{K}_n = \frac{\boldsymbol{P}_n^-\boldsymbol{h}}{\boldsymbol{h}^T\boldsymbol{P}_n^-\boldsymbol{h} + r_n} \qquad \begin{aligned} \hat{\boldsymbol{x}}_n &= \hat{\boldsymbol{x}}_n^- + \boldsymbol{K}_n[y_n - \boldsymbol{h}^T\hat{\boldsymbol{x}}_n^-] \\ \boldsymbol{P}_n &= \boldsymbol{P}_n^- - \boldsymbol{K}_n\boldsymbol{h}^T\boldsymbol{P}_n^- \end{aligned}$$

where $q_n = \mathbb{E}\{w_n^2\}$, $r_n = \mathbb{E}\{v_n^2\}$, $\hat{\boldsymbol{x}}_n^- = \hat{E}\{\boldsymbol{x}_n | y_{n-1}, \ldots, y_1\}$ (*prior* estimate of $\boldsymbol{x}_n$), and $\hat{\boldsymbol{x}}_n = \hat{E}\{\boldsymbol{x}_n | y_n, \ldots, y_1\}$ (*posterior* estimate of $\boldsymbol{x}_n$).

Kalman filters for powerline noise estimation and removal from biosignals (continued)

Powerline noise removal from the ECG [Sameni, 2012]

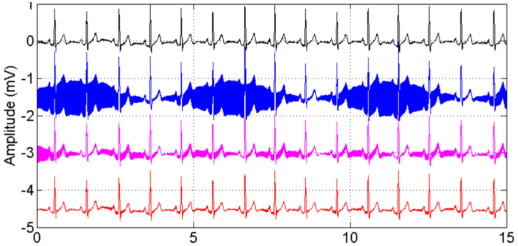


Figure: From top to bottom: a) a clean ECG, b) diluted with nonstationary powerline, c) denoised using a second-order IIR notch filter, and d) denoised using the Kalman filter

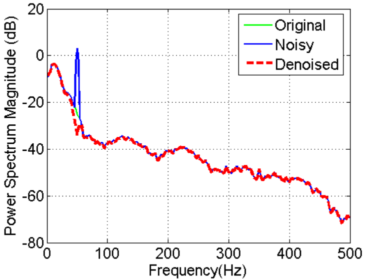


Figure: ECG spectrum before and after powerline cancellation with the Kalman filter

Kalman filters for powerline noise estimation and removal from biosignals (continued)

Powerline noise removal from the EEG [Sameni, 2012]

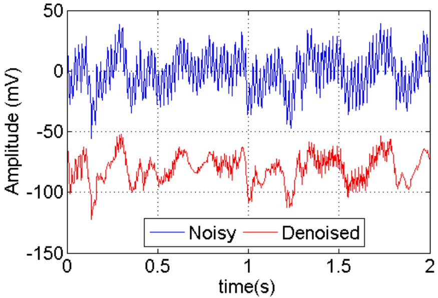


Figure: a) EEG diluted with powerline, b) denoised using the Kalman filter

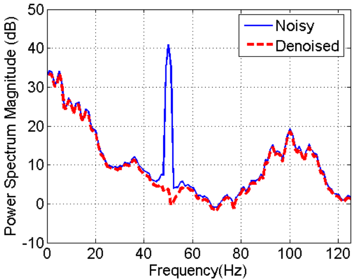


Figure: EEG spectrum before and after powerline cancellation with the Kalman filter

Kalman filters for powerline noise estimation and removal from biosignals (continued)

How does it work?

The Kalman notch filter response in steady state (**Wiener filter**) [Sameni, 2012]:

$$G(z) = \frac{\alpha(1 - 2\cos(\omega_0)z^{-1} + z^{-2})}{1 - \cos(\omega_0)\left(\frac{4\alpha}{\alpha+1}\right)z^{-1} + \alpha z^{-2}}$$

α: a function of the q_n/r_n ratio.

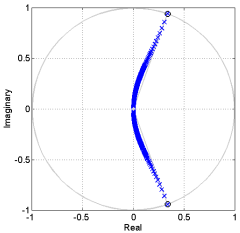


Figure: Root loci of the KF subject to variations of α

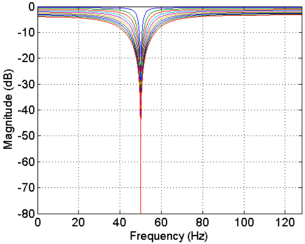


Figure: KF frequency response with variation of α.

Interpretation: The KF changes its frequency response in real-time to adapt itself to the signal-noise contents of the observations.

Further reading on powerline cancellation using Kalman filters

- B. L. Scala and R. Bitmead. Design of an extended Kalman filter frequency tracker. *IEEE Transactions on Signal Processing*, 44(3):739–742, Mar. 1996. doi: 10.1109/78.489052
- L. D. Avendaño-Valencia, L. E. Avendaño, J. Ferrero, and Castellanos-Domínguez. Improvement of an extended Kalman filter power line interference suppressor for ECG signals. In *2007 Computers in Cardiology*, pages 553 –556. IEEE, Sept. 2007. doi: 10.1109/cic.2007.4745545
- R. Sameni. A Linear Kalman Notch Filter for Power-Line Interference Cancellation. In *Proceedings of the 16th CSI International Symposium on Artificial Intelligence and Signal Processing (AISP)*, pages 604–610, Shiraz, Iran, 2-3 May 2012 2012. URL `https://doi.org/10.1109/AISP.2012.6313817`
- G. J. J. Warmerdam, R. Vullings, L. Schmitt, J. O. E. H. V. Laar, and J. W. M. Bergmans. A Fixed-Lag Kalman Smoother to Filter Power Line Interference in Electrocardiogram Recordings. *IEEE Transactions on Biomedical Engineering*, 64(8):1852–1861, Aug. 2017. doi: 10.1109/tbme.2016.2626519

Outline

Linear Kalman filters for autoregressive models

Exercise 1:

A very coarse stochastic model of biological events such as background EEG (non-evoked potentials), heart-rate time-series, dipolar movements, etc. is a first-order autoregressive (AR) model:

$$s_n = \alpha s_{n-1} + w_n$$
$$x_n = s_n + v_n$$

where s_n is the biological event of interest, v_n is the background event, x_n is the measured signal, and $0 < \alpha < 1$ controls the spectral shape of the desired signal.

1. Write the standard Kalman filter equations for estimation of s_n.
2. Assuming stationary process-noise $q = \mathbb{E}\{w_n^2\}$ and $r = \mathbb{E}\{v_n^2\}$, derive a closed-form equation for the system with input x_n and output $\hat{v}_n = x_n - \hat{s}_n$, when the KF reaches steady-state.
3. What is the transfer function of the above system and how does it relate to q and r? How many parameters does the filter have?

Question: Can you extend the results to higher-order AR models?

Linear Kalman filters for sleep staging

- The spectrum of the EEG varies at different levels of consciousness and sleep.
- A suitable approach for modeling EEG nonstationarities is by using a time-variant autoregressive (TVAR) random process:

$$x_n = \sum_{k=1}^{p} a_n^{(k)} x_{n-k} + v_n$$

- The state-space form of a TVAR is:

$$\mathbf{s}_{n+1} = \mathbf{s}_n + \mathbf{w}_n$$

$$x_n = \mathbf{h}_n^T \mathbf{s}_n + v_n$$

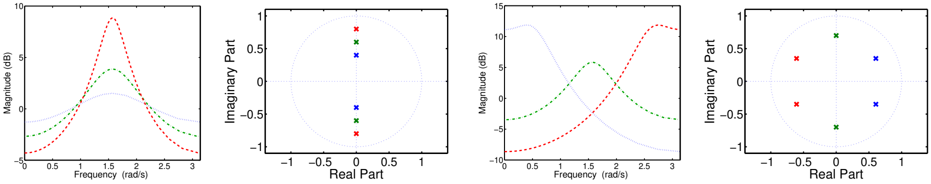
where $\mathbf{s}_n = (a_n^{(1)}, \ldots, a_n^{(p)})^T$ and $\mathbf{h}_n = (y_{n-1}, \ldots, y_{n-p})^T$.

Linear Kalman filters for sleep staging
(continued)

- TVAR models with slow variations are equivalent to the following transfer function:

$$H(z;n) = \frac{X(z)}{V(z)} \approx \frac{1}{1 - a_n^{(1)} z^{-1} - \dots a_n^{(p)} z^{-p}}$$

- Specifically, for $p = 2$, the system has two time-variant poles with center frequency f_n and bandwidth bw_n as functions of the TVAR model coefficients $(a_n^{(1)}, a_n^{(2)})$.



(a) (b) (c) (d)

Figure: TVAR(2) model pole loci vs filter characteristics

Linear Kalman filters for sleep staging
(continued)

Sleep staging results from [Rahbar Alam and Sameni, 2020]:

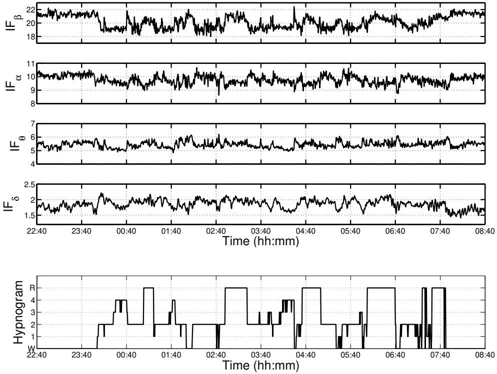


Figure: a) The estimated IF different EEG bandwidths vs, b) the corresponding Hypnogram index

Further reading on Kalman filters for EEG applications

- L. Patomaki, J. Kaipio, and P. Karjalainen. Tracking of nonstationary EEG with the roots of ARMA models.
  In *Proceedings of 17th International Conference of the Engineering in Medicine and Biology Society*, volume 2, pages 877–878 vol.2, 1995.
  doi: 10.1109/IEMBS.1995.579249

- A. H. Omidvarnia, M. Mesbah, M. S. Khlif, J. M. O'Toole, P. B. Colditz, and B. Boashash. Kalman filter-based time-varying cortical connectivity analysis of newborn EEG.
  In *2011 Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, pages 1423–1426, 2011.
  doi: 10.1109/IEMBS.2011.6090335

- V. Oikonomou, A. Tzallas, and D. Fotiadis. A Kalman filter based methodology for EEG spike enhancement.
  Computer Methods and Programs in Biomedicine, 85(2):101–108, 2007.
  ISSN 0169-2607.
  doi: https://doi.org/10.1016/j.cmpb.2006.10.003

- M. J. Barton, P. A. Robinson, S. Kumar, A. Galka, H. F. Durrant-Whyte, J. Guivant, and T. Ozaki. Evaluating the Performance of Kalman-Filter-Based EEG Source Localization.
  IEEE Transactions on Biomedical Engineering, 56(1):122–136, 2009.
  doi: 10.1109/TBME.2008.2006022

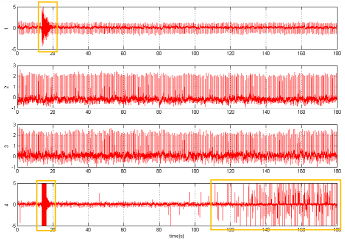
Outline

Robust heart-rate tracking with Kalman filters

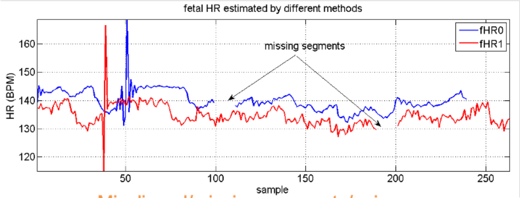
Robust heart rate estimation from multiple noisy sources using a Kalman filter [Li et al., 2007]

$$\mathrm{HR}[n] = \frac{\sigma_2[n]^2}{\sigma_1[n]^2 + \sigma_2[n]^2}\mathrm{HR}_1[n] + \frac{\sigma_1[n]^2}{\sigma_1[n]^2 + \sigma_2[n]^2}\mathrm{HR}_2[n]$$

Applications in multichannel fetal heart-rate tracking



Missing (or noisy) segment recovery



Misaligned/missing segments/noisy

Dynamic modeling of the ECG

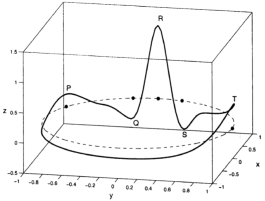
McSharry-Clifford's synthetic ECG model in Cartesian coordinates [McSharry et al., 2003]

$$\begin{cases} \dot{x} = \rho x - \omega y \\ \dot{y} = \rho y + \omega x \\ \dot{z} = -\sum_{i\in\{P,Q,R,S,T\}} a_i \Delta\theta_i \exp(-\frac{\Delta\theta_i^2}{2b_i^2}) - (z - z_0) \end{cases}$$

- (x, y, z) is the state vector, $\rho = 1 - \sqrt{x^2 + y^2}$, $\Delta\theta_i = (\theta - \theta_i) \mod (2\pi)$, $\theta = \text{atan2}(y, x)$, and ω is the angular velocity of the trajectory as it moves around the limit cycle in the $x - y$ plane.
- a_i, b_i, and θ_i are amplitudes, widths, and centers parameters of the Gaussian terms.

Table: Parameters of the synthetic ECG model

Index(i)	P	Q	R	S	T
$\theta_i(rads.)$	$-\pi/3$	$-\pi/12$	0	$\pi/12$	$\pi/2$
a_i	1.2	-5.0	30	-7.5	0.75
b_i	0.25	0.1	0.1	0.1	0.4

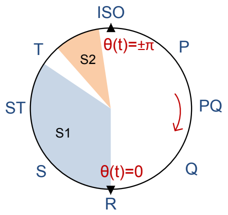


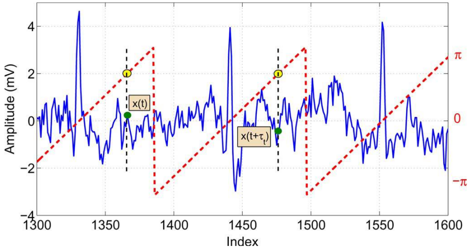
The cardiac phase signal

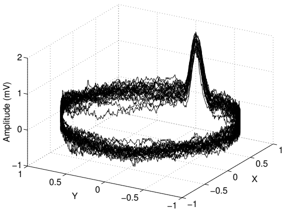
Cardiac phase [Sameni et al., 2008a]

An abstract phase signal $\theta(t) \in [-\pi, \pi]$ introduced for alignment of identical time instants during the cardiac cycle, with the following properties:

1. The R-peak is considered at $\theta(t) = 0$
2. $\theta(t_1) = \theta(t_2) \iff t_1,\ t_2$ correspond to identical depolarization/repolarization times during the cardiac cucle







Dynamic modeling of the ECG
(continued)

Modified McSharry-Clifford's model in polar coordinates [Sameni et al., 2005]

$$\dot{\theta}(t) = \omega(t)$$

ECG phase dynamics

$$\dot{s}(t) = -\sum_i \frac{\alpha_i \omega(t)}{b_i^2} \Delta\theta_i(t) \exp[-\frac{\Delta\theta_i(t)^2}{2b_i^2}]$$

ECG morphology dynamics

$$x(t) = s(t) + v(t)$$

Noisy observations

$\omega(t) = 2\pi \times \mathrm{HR}(t)$, $\Delta\theta_i(t) = [\theta(t) - \theta_i] \mod (2\pi)$ and $\mathrm{HR}(t)$ is the beat-wise heart-rate in Hz (beats per second).

The above dynamic form can be used in an **extended Kalman filter (EKF)** for ECG denoising.

An extended Kalman filter for ECG denoising

Removing white noise from the ECG using an EKF [Sameni et al., 2007]

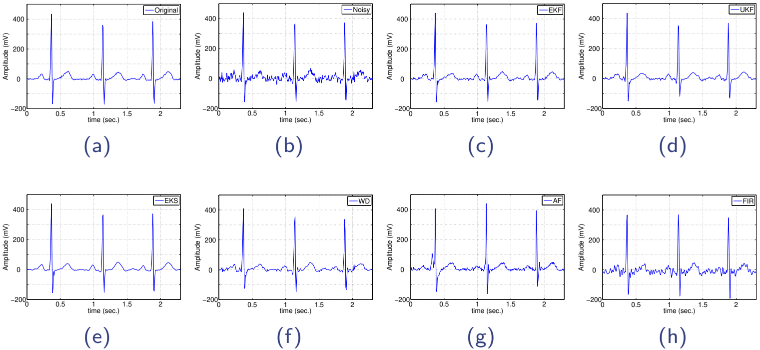
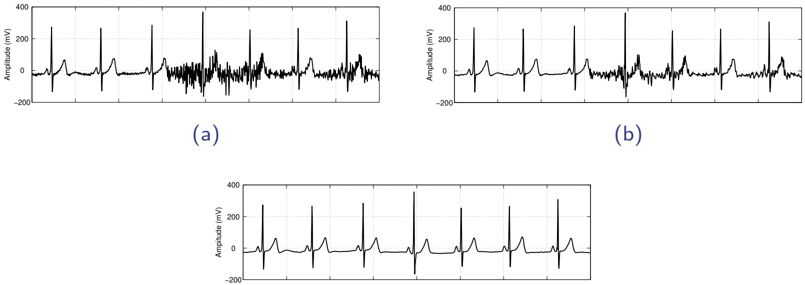


Figure: a) Original, b) Noisy (SNR=6 dB), c) EKF, d) UKF, e) EKS, f) WD, g) AF, h) FIR

An extended Kalman filter for ECG denoising
(continued)

Removing colored noise from the ECG using an EKF [Sameni et al., 2007]



(a) (b) (c)

Figure: Muscle artifact removal: (a) Noisy ECG with SNR=6dB (b) EKS result without measurement noise variance adaptation (c) EKS result using measurement noise variance adaptation

Dynamic modeling of noninvasive fetal electrocardiogram

A dynamic model for maternal abdominal recordings [Sameni, 2008]

- *Process equations*:

$$\theta(t+1) = [\theta(t) + \omega_m(t)] \mod (2\pi)$$
$$s_m(t+1) = s_m(t) - \omega_m(t) \sum_{i=1}^{k} \frac{\alpha_i \tilde{\theta}_i(t)}{b_i^2} \exp(\frac{-\tilde{\theta}_i(t)^2}{2b_i^2}) + w(t)$$

- *Observation equations*:

$$\phi(t) = \theta(t) + \nu(t)$$
$$x(t) = s_m(t) + s_f(t) + \eta(t)$$

$\tilde{\theta}_i(t) = [\theta(t) - \theta_i] \mod (2\pi)$, $\omega_m(t) = 2\pi f_m(t)/f_s$: normalized mECG angular velocity, $f_m(t)$: instantaneous maternal heart-rate, f_s: sampling frequency, $(\alpha_i, b_i, \theta_i)$: amplitude, width and center of Gaussian kernels, $[\theta(t), s_m(t)]^T$: state vector, $\phi(t)$: maternal cardiac phase measurement, $x(t)$: maternal abdominal ECG measurement, $w(t)$: process noise, $\nu(t)$: phase measurement noise, and $\eta(t)$: ECG measurement noise.

The above dynamic form can be used in an **extended Kalman filter (EKF)** for ECG denoising.

Dynamic modeling of noninvasive fetal electrocardiogram (continued)

Maternal ECG cancellation from abdominal leads

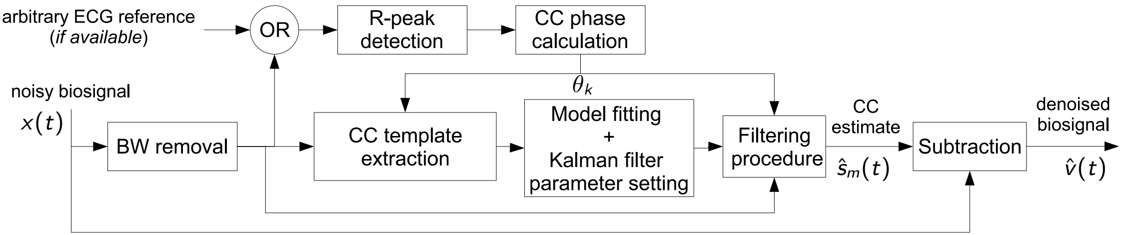


Figure: Maternal ECG cancellation using an EKF [Sameni et al., 2008b]

Simultaneous maternal-fetal ECG Kalman filtering

The maternal and fetal ECG cardiac phase signals

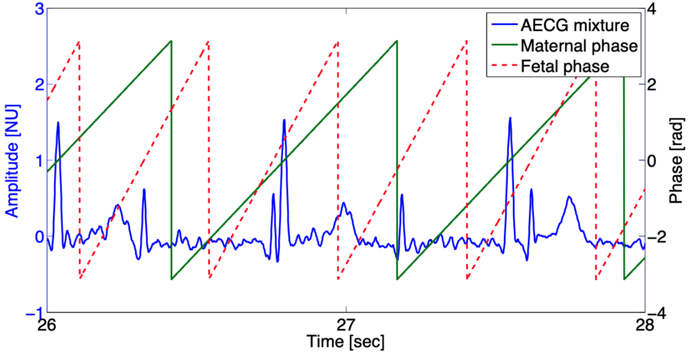


Figure: Maternal abdominal ECG and the cardiac phase signals of the mother and fetus; adopted with permission from [Behar, 2014]

Simultaneous maternal-fetal ECG Kalman filtering
(continued)

Augmented maternal-fetal dynamic model for maternal abdominal recordings

- *Process equations*:

$$
\begin{aligned}
&\theta_m(t+1) = [\theta_m(t) + \omega_m(t)] \mod (2\pi)\\
&\theta_f(t+1) = [\theta_f(t) + \omega_f(t)] \mod (2\pi)\\
&s_m(t+1) = s_m(t) - \omega_m(t)\sum_{i=1}^{k} \frac{\alpha_{mi}\tilde{\theta}_{mi}(t)}{b_{mi}^2}\exp\left(\frac{-\tilde{\theta}_{mi}(t)^2}{2b_{mi}^2}\right) + w_m(t)\\
&s_f(t+1) = s_f(t) - \omega_f(t)\sum_{i=1}^{k} \frac{\alpha_{fi}\tilde{\theta}_{fi}(t)}{b_{fi}^2}\exp\left(\frac{-\tilde{\theta}_{fi}(t)^2}{2b_{fi}^2}\right) + w_f(t)
\end{aligned}
$$

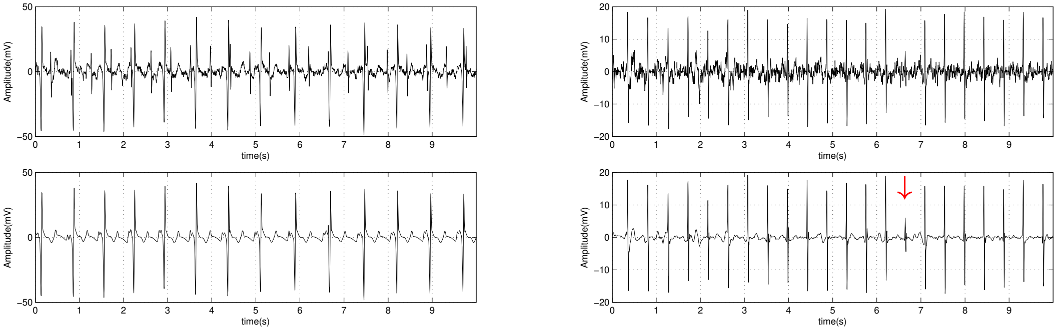
- *Observation equations*:

$$
\begin{aligned}
&\phi_m(t) = \theta_m(t) + \nu_m(t)\\
&\phi_f(t) = \theta_f(t) + \nu_f(t)\\
&x(t) = s_m(t) + s_f(t) + \eta(t)
\end{aligned}
$$

*The notations are extensions of the previous maternal model to both the mother (m) and the fetus (f).

Dynamic modeling of noninvasive fetal electrocardiogram
(continued)

Sample results



(Top-left) a channel of the DaISy dataset abdominal lead [De Moor, 1997], (bottom-left) mECG EKF estimate, (top-right) residual fECG after mECG removal, (bottom-right) fECG after denoising with EKF [Sameni, 2008].

Dynamic modeling of noninvasive fetal electrocardiogram (continued)

FECG denoising Kalman filtering with confidence intervals

- Estimation covariance matrices ($\boldsymbol{P}_n^-$ and $\hat{\boldsymbol{P}}_n$) are inherent parts of the Kalman filtering process.
- For a well-functioning Kalman filter, $\boldsymbol{P}_n^-$ and $\hat{\boldsymbol{P}}_n$ can be used to derive confidence intervals for the estimated signals, which is a very practical feature of the Kalman filter.

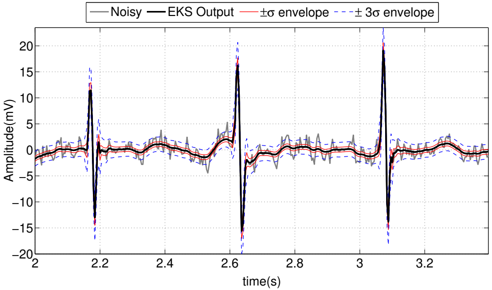


Figure: fECG before and after the EKF filtering and its $\pm\sigma$ and $\pm 3\sigma$ confidence intervals [Sameni, 2008]

Alternative dynamic forms for adult-fetal ECG modeling

The polar form of the McSharry-Clifford model of the ECG can be stated in alternative forms.

An alternative dynamic ECG model [Kheirati Roonizi, 2017]

$$\dot{\theta}(t) = \omega$$
$$\dot{\phi}_i(t) = -\omega \left(\frac{\theta(t) - \theta_i}{b_i^2} \right) \phi_i(t) \quad (i = 1, \ldots, N)$$
$$x(t) = \sum_{i=1}^{N} \phi_i(t)$$

Multichannel fetal ECG processing

Is single-channel adult/fetal ECG modeling and processing accurate enough? **Not quite!**

- The heart is a spatially distributed source $\rightarrow$ multichannel modeling/processing does a better job
- The fetus moves and rotates $\rightarrow$ difficult to find a single abdominal lead for fECG acquisition
- Maternal abdominal leads are noisy $\rightarrow$ multichannel redundancy improves the performance
- Maternal and fetal QRS overlap (in time and frequency) $\rightarrow$ an alternative modality (space) is required to separate them from one another

Solution: a unified framework for spatio-temporal filtering

Combining temporal estimation and filtering techniques (conventional filters, adaptive filters, Kalman filter, wavelets, etc.) with blind and semi-blind source separation techniques (PCA, ICA, πCA, etc.)

Multichannel ECG filtering

Multichannel vectorcardiogram model [Sameni et al., 2006]

$$\dot{\theta}(t) = \omega(t)$$

ECG phase

$$\dot{s}_1(t) = -\sum_i \frac{\alpha_{1i}\omega(t)}{b_{1i}^2}\Delta\theta_{1i}(t)\exp[-\frac{\Delta\theta_{1i}(t)^2}{2b_{1i}^2}]$$

1st VCG dimension (in source space)

$$\vdots$$

$$\dot{s}_n(t) = -\sum_i \frac{\alpha_{ni}\omega(t)}{b_{ni}^2}\Delta\theta_{ni}(t)\exp[-\frac{\Delta\theta_{ni}(t)^2}{2b_{ni}^2}]$$

n'th VCG dimension (in source space)

$$\boldsymbol{x}(t) = \boldsymbol{H}\boldsymbol{s}(t) + \boldsymbol{v}(t)$$

Multi-channel noisy observations

where $\Delta\theta_{ni}(t) = [\theta(t) - \theta_{ni}] \mod (2\pi)$, $\boldsymbol{s}(t) = [s_1(t), \ldots, s_n(t)]^T$ is the vector of modeled VCG dimensions (in source space), $\boldsymbol{H}$ is a mixing matrix, found by **principal component analysis (PCA)** or **independent component analysis (ICA)**.

Spatio-temporal filtering of the fetal ECG

A spatio-temporal approach to fECG extraction and filtering

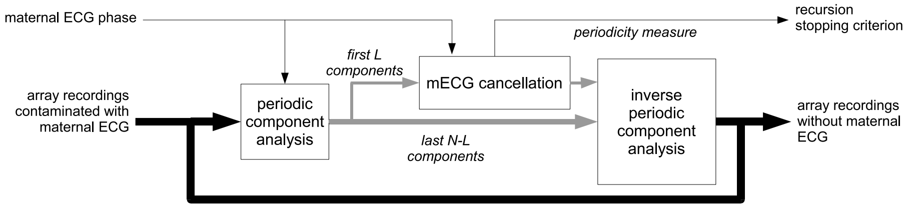


Figure: A DEFL algorithm for fECG extraction in highly noisy and rank-deficient scenarios [Sameni et al., 2010]

Intuition:

Per-iteration (i), it performs a spatio-temporal shrinkage (a spatio-temporal version of wavelet shrinkage) [Sameni, 2021a]:

$$\boldsymbol{y}_i(t) = \boldsymbol{W}_i^{-T}\boldsymbol{G}\big(\boldsymbol{W}_i^{T}\boldsymbol{x}_i(t); L\big)$$

Outline

Linear Kalman filters for PCG processing

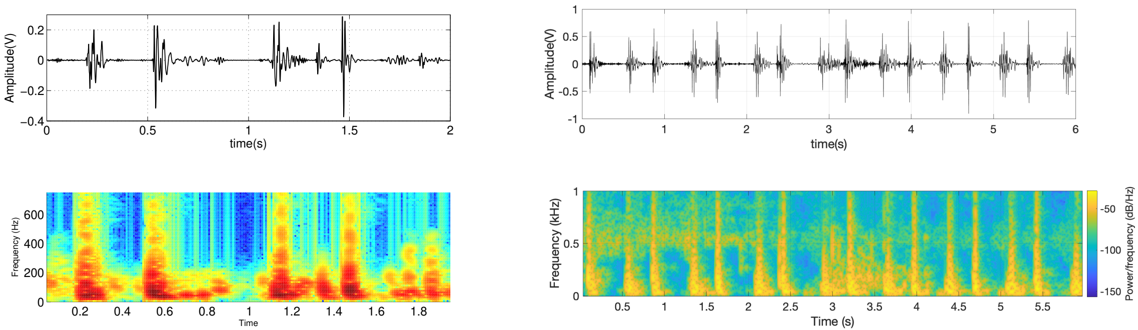


Figure: Typical adult PCG (left) and fetal PCG (right) and their spectrograms

- The phonocardiogram (PCG) components can be modeled with a damped oscillation.
- Tracking the parameters of this model can be used to denoise the PCG in background audio noise and to identify the major PCG components (S1 and S2).

Linear Kalman filters for PCG processing
(continued)

PCG data model [Kheram, 2019]

The PCG can be considered as the output of a second-order **innovation filter** with frequency response:

$$G(z) = k\frac{1-\alpha\cos(\omega_0)z^{-1}}{1-2\alpha\cos(\omega_0)z^{-1}+\alpha^2z^{-2}}$$

Or equivalently the impulse response:

$$g_n = k\alpha^n\cos(\omega_0 n)u[n]$$

where $\omega_0 = 2\pi f_0/f_s$, f_0 is the center frequency of the filter, f_s is the sampling frequency, α $(0<\alpha<1)$ controls the filter bandwidth, and k is the filter gain.

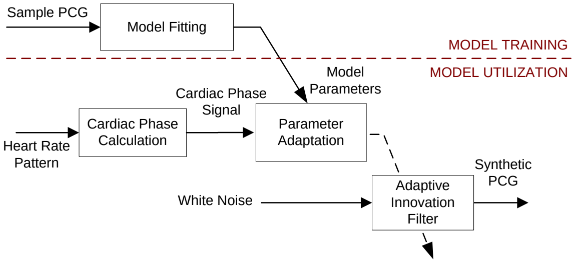


Figure: The overall scheme of the adaptive innovation filter proposed for PCG modeling

Linear Kalman filters for PCG processing
(continued)

Data model analysis

$$x_n = 2\alpha\cos(\omega_0)x_{n-1} - \alpha^2 x_{n-2} + kw_n - k\alpha\cos(\omega_0)w_{n-1} \tag{1}$$

where w_n is zero-mean white noise with variance $q_n^2 = \mathbb{E}\{x_n^2\}$.

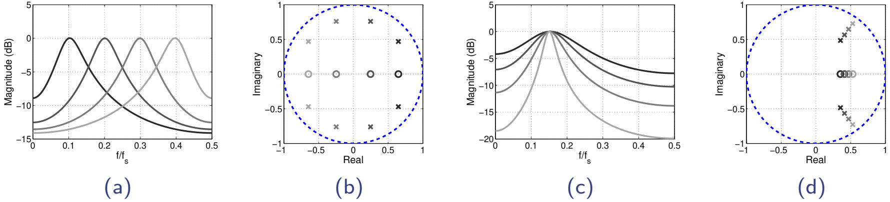


Figure: Frequency responses and pole-zero diagrams of the innovation filter. The system has two complex conjugate poles and a real zero in all cases. From dark to light, (a) and (b) correspond to $\alpha = 0.8$ and $\omega_0 = 0.2\pi, 0.4\pi, 0.6\pi, 0.8\pi$. From dark to light, (c) and (d) correspond to $\omega_0 = 0.3\pi$ and $\alpha = 0.6, 0.7, 0.8, 0.9$.

Linear Kalman filters for PCG processing
(continued)

A useful second-order state-space form [Gordon and Smith, 1985]

- Signals with mono-modal spectra can be stated in state-space form:

$$\begin{aligned} c_n &= \alpha c_{n-1} - \alpha\epsilon d_{n-1} + w_n \\ d_n &= \alpha\epsilon c_{n-1} + \alpha(1-\epsilon^2) d_{n-1} \\ x_n &= c_n + v_n \end{aligned}$$

$\epsilon \stackrel{\Delta}{=} \sqrt{2}\sin(\omega_0/2)$, $\boldsymbol{s}_n = (c_n \quad d_n)^T$ is the state vector, and v_n is the background noise.

The state-space form can be used in a Kalman filter for adult/fetal PCG denoising [Kheram, 2019].

Question: : Is the dynamic model for a second-order model unique?

- No! For example, the second-order dynamic model used earlier for powerline noise model can also be used for PCG modeling.
- KF stability and performance aspects also impact the dynamic model selection.

Outline

Compartmental modeling

Definition

A **compartmental model** is a **weighted directed graph** representation of a linear or nonlinear **dynamic system**

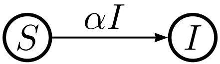
Representing systems as compartmental models

1. Partition the population into **homogeneous** groups of individuals (known as **compartments**)
2. Assign states (time-variables) to each compartment
3. Write flow equations between the compartments
4. Model analysis, simulation, parameter estimation, and prediction

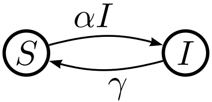
Examples of compartmental models for contagious diseases

The susceptible-infected models with and without immunity

The population is split to susceptibles $S(t)$ and infected $I(t)$ individuals, subject to $S(t) + I(t) = N$



$$\dot{S}(t) = -\alpha S(t)I(t)$$
$$\dot{I}(t) = \alpha S(t)I(t)$$

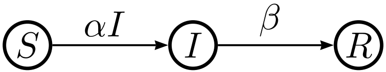


$$\dot{S}(t) = -\alpha S(t)I(t) + \gamma I(t)$$
$$\dot{I}(t) = \alpha S(t)I(t) - \gamma I(t)$$

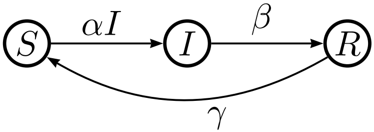
Examples of compartmental models for contagious diseases
(continued)

The susceptible-infected-recovered (SIR) model

The population is divided to three groups: $S(t)$: susceptibles, $I(t)$: infected, $R(t)$: recovered, subject to $S(t) + R(t) + I(t) = N$.



$$\dot{S}(t) = -\alpha S(t)I(t)$$
$$\dot{I}(t) = \alpha S(t)I(t) - \beta I(t)$$
$$\dot{R}(t) = \beta I(t)$$

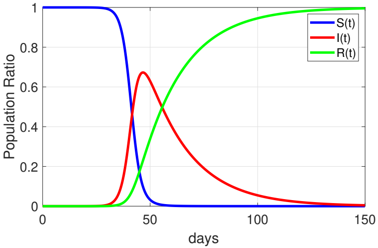


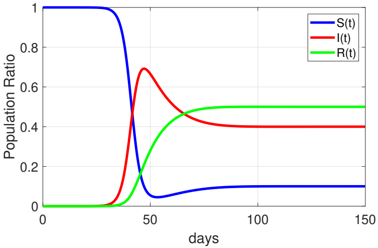
$$\dot{S}(t) = -\alpha S(t)I(t) + \gamma R(t)$$
$$\dot{I}(t) = \alpha S(t)I(t) - \beta I(t)$$
$$\dot{R}(t) = \beta I(t) - \gamma R(t)$$

Examples of compartmental models for contagious diseases
(continued)

Typical SIR solutions without immunity

With a given initial condition SIR models can be numerically solved.



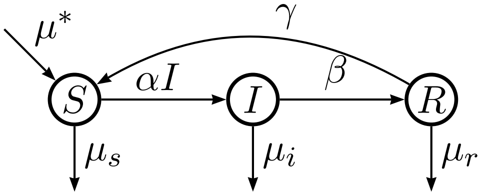


✎ For epidemic diseases, the peak and slope of $I(t)$ is more important than the total number of infected individuals. Why?

Examples of compartmental models for contagious diseases
(continued)

A fatal SIR model without life-time immunity

The family of SIR models may also have birth and death rates (making the system *open*):



Compartmental modeling of epidemic diseased

Epidemic disease spread in large populations

A population of N individuals can be partitioned into **population fractions**:

- **Susceptibles**: $s(t)$
- **Exposed** (without symptoms): $e(t)$
- **Infected** (with symptom): $i(t)$
- **Recovered**: $r(t)$
- **Deceased**: $p(t)$

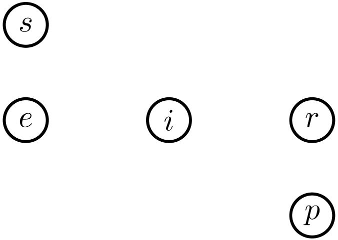
subject to $s(t) + e(t) + i(t) + r(t) + p(t) = 1$

✎ In the sequel we study a: **mortal non-immunizing susceptible-exposed-infected-recovered** (SEIR) model

Compartmental modeling of epidemic diseased
(Continued)

A mortal SEIR model without lifetime immunity

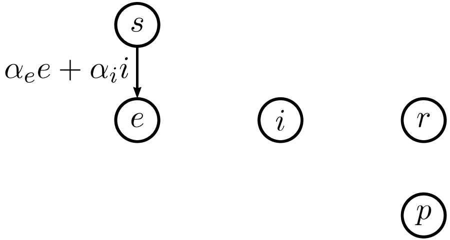
Start with the compartments.



Compartmental modeling of epidemic diseased
(Continued)

A mortal SEIR model without lifetime immunity

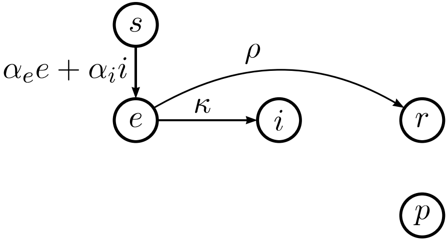
Individuals are infected as they contact infected or exposed subjects.



Compartmental modeling of epidemic diseased
(Continued)

A mortal SEIR model without lifetime immunity

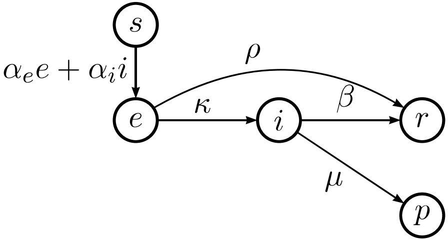
The exposed either recover or become infected with symptoms.



Compartmental modeling of epidemic diseased
(Continued)

A mortal SEIR model without lifetime immunity

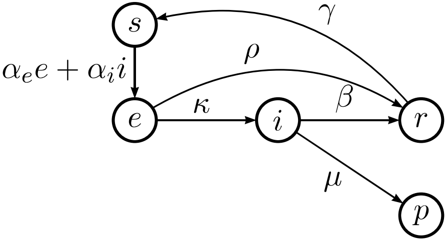
The infected either recover or pass away.



Compartmental modeling of epidemic diseased
(Continued)

A mortal SEIR model without lifetime immunity

The recovered may again become susceptible.

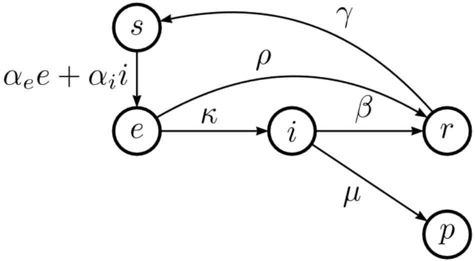


Compartmental modeling of epidemic diseased

(Continued)

A non-immunizing fatal susceptible-exposed-infected-recovered (SEIR) model [Sameni, 2020]

The full model and its corresponding dynamic equations, which can be numerically solved from **initial conditions** $[s(t_0), e(t_0), i(t_0), r(t_0), p(t_0)]$:



$$\frac{ds(t)}{dt} = -\alpha_e s(t)e(t) - \alpha_i s(t)i(t) + \gamma r(t)$$

$$\frac{de(t)}{dt} = \alpha_e s(t)e(t) + \alpha_i s(t)i(t) - \kappa e(t) - \rho e(t)$$

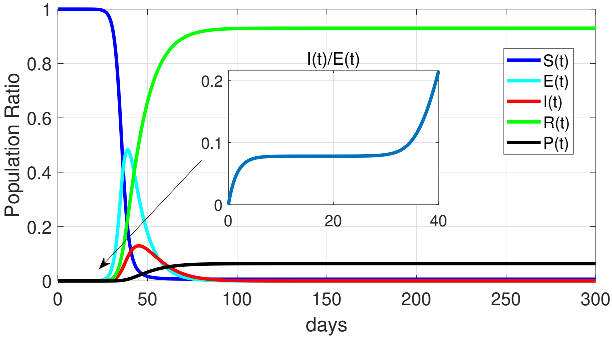
$$\frac{di(t)}{dt} = \kappa e(t) - \beta i(t) - \mu i(t)$$

$$\frac{dr(t)}{dt} = \beta i(t) + \rho e(t) - \gamma r(t)$$

$$\frac{dp(t)}{dt} = \mu i(t)$$

Typical deterministic solutions

The life-time immune case ($\gamma = 0$)



Compartmental modeling of epidemic diseased

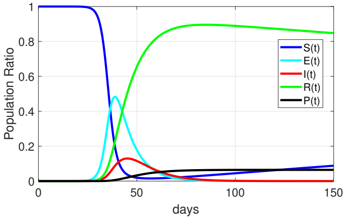
(Continued)

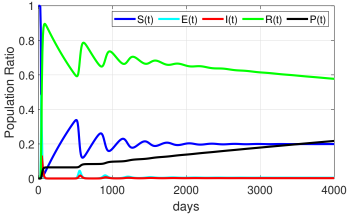
What can we learn from the macro-level SEIR model?

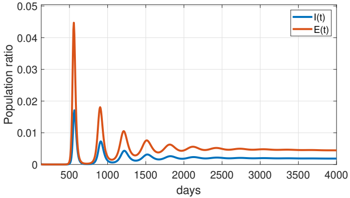
1. Is the model **observable**? Are its parameters **identifiable**?
2. Is the situation **stable**?
3. The future trend of the pandemic can be **estimated** with quantitative **confidence intervals**
4. The peak of the infected population (to avoid reaching the healthcare system break-point) can be estimated
5. The effect of quarantine, social distancing, lockdown, and reopening can be studied
6. Potential future outbreaks
7. The model accuracy and the consistency of the reported data

Typical solutions
(Continued)

The non-immunizing case ($\gamma \neq 0$) in short- and long-term

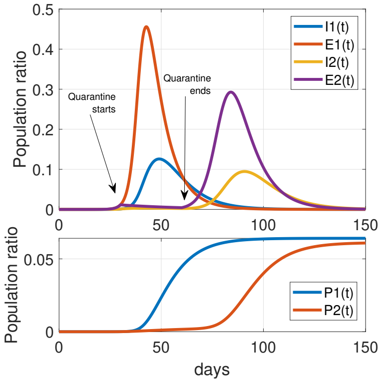






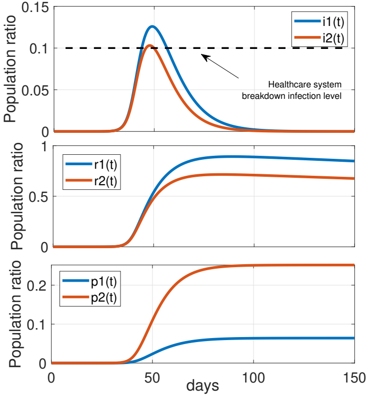
Typical solutions
(continued)

Insufficient lockdown periods



Typical solutions
(continued)

Healthcare system saturation and breakdown



Time-variant model parameters:

$$\beta(t) = (\beta_s - \beta_0)h(i(t)) + \beta_0$$
$$\mu(t) = (\mu_s - \mu_0)h(i(t)) + \mu_0$$

where

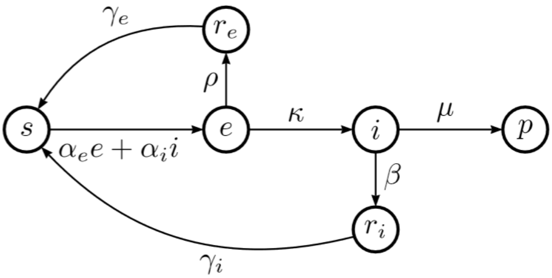
$$h(i) = \frac{1}{2}[1 + \tanh\left(\frac{i - i_0}{\sigma}\right)]$$

Model extensions

Is the model unique? **No!**

A modified SEIR model [Sameni, 2020]

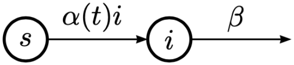
- An extension of the fatal SEIR model for Coronavirus modeling is to assume that the recoveries from exposure and infection are separate compartments r_e and r_i:
- **Other possible extensions**: consider age groups, gender, seasons, geopolitical factors (hemispheres, cities, countries, continents, etc.)



Pandemic trend forecasting using Kalman filters

Data model

$$\dot{s}(t) = -\alpha(t)s(t)i(t)$$
$$\dot{i}(t) = \alpha(t)s(t)i(t) - \beta i(t)$$
$$\dot{\alpha}(t) = -\gamma\alpha(t) + \gamma h[\mathbf{u}(t)]$$



- $s(t)$: susceptibles
- $i(t)$: infectious
- $\alpha(t)$: contact rate
- β: recovery/quarantine rate
- γ: NPI to effect rate
- $u(t)$: non-pharmaceutical interventions (NPI)
- $h[\cdot]$: NPI to contact map

Pandemic prediction by extended Kalman filtering and finite-interval extended Kalman smoother

State equations:

$$\dot{s}(t) = -\alpha(t)s(t)i(t) + w_s(t)$$
$$\dot{i}(t) = \alpha(t)s(t)i(t) - \beta i(t) + w_i(t)$$
$$\dot{\alpha}(t) = -\gamma\alpha(t) + \gamma h[\mathbf{u}^*(t)] + w_\alpha(t)$$

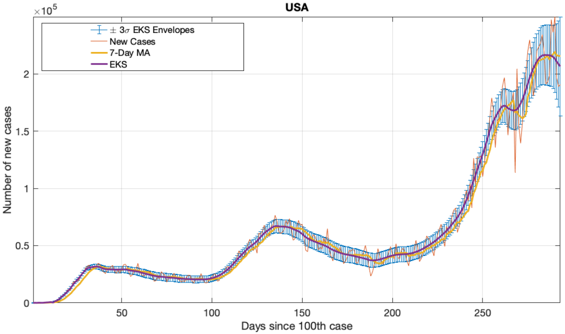
Observation equations:

$$\text{New cases: } n(t) = \alpha(t)s(t)i(t) + v(t)$$

$$\text{Total confirmed cases: } c(t) = s(t_0) - s(t) + v(t)$$

Pandemic prediction by extended Kalman filtering and finite-interval extended Kalman smoother

Sample prediction result for the US new cases



Pandemic control through non-pharmaceutical interventions

Bi-objective cost functions

Human cost:

$$J_0(\mathbf{u}) = \int_{t=t_0}^{t_1} \alpha(t)s(t)i(t)dt,$$

Intervention cost:

$$J_1(\mathbf{u}) = \int_{t=t_0}^{t_1} \mathbf{w}(t)^T\mathbf{u}(t)dt$$

Bi-objective cost:

$$J(\mathbf{u}) = (1-\epsilon)J_0(\mathbf{u}) + \epsilon J_1(\mathbf{u}) \quad \text{s.t. } \mathbf{u} \in \Gamma$$

Admissible NPI set:

$$\Gamma = \{\mathbf{u} | \mathbf{u}^{\min} \leq \mathbf{u}(t) \leq \mathbf{u}^{\max}, \forall t \in [t_0, t_1]\}$$

Optimal cost:

$$J(\mathbf{u}^*) = \min_{\Gamma}(J(\mathbf{u}))$$

Pandemic control through non-pharmaceutical interventions
(continued)

A finite-horizon optimal control strategy [Sameni, 2021b]

The Hamiltonian:

$$\begin{aligned}\mathcal{H} = \quad & (1-\epsilon)\alpha(t)s(t)i(t) + \epsilon\mathbf{w}(t)^T\mathbf{u}(t)\\ & -\lambda_1(t)\alpha(t)s(t)i(t)\\ & +\lambda_2(t)[\alpha(t)s(t)i(t) - \beta i(t)]\\ & -\gamma\lambda_3(t)\{\alpha(t) - h[\mathbf{u}(t)]\}\end{aligned}$$

Pontryagin's minimum principle requires:

$$\begin{aligned}\dot{\lambda}_1(t) &= -\frac{\partial\mathcal{H}}{\partial s} = [\lambda_1(t) - \lambda_2(t) - (1-\epsilon)]\alpha(t)i(t)\\ \dot{\lambda}_2(t) &= -\frac{\partial\mathcal{H}}{\partial i} = [\lambda_1(t) - \lambda_2(t) - (1-\epsilon)]\alpha(t)s(t) + \beta\lambda_2(t)\\ \dot{\lambda}_3(t) &= -\frac{\partial\mathcal{H}}{\partial \alpha} = [\lambda_1(t) - \lambda_2(t) - (1-\epsilon)]s(t)i(t) + \gamma\lambda_3(t)\\ \mathcal{H}(\mathbf{u}^*) &\leq \mathcal{H}(\mathbf{u}), \quad \forall \mathbf{u} \in \Gamma\end{aligned}$$

Pandemic control through non-pharmaceutical interventions (continued)

Optimal NPI for sample contact rate maps [Sameni, 2021b]

1. **Linear (LASSO) regression**: $h[\mathbf{u}(t)] = b + \mathbf{a}^T[\mathbf{u}^{\max} - \mathbf{u}(t)]$

$$u_k^*(t) = \begin{cases} u_k^{\min} : & \epsilon w_k(t) > \gamma\lambda_3(t) a_k \\ u_k^{\max} : & \epsilon w_k(t) < \gamma\lambda_3(t) a_k \end{cases}$$

2. **Quadratic regression**: $h[\mathbf{u}(t)] = b + \mathbf{a}^T[\mathbf{u}^{\max} - \mathbf{u}(t)] + \frac{1}{2}[\mathbf{u}^{\max} - \mathbf{u}(t)]^T\mathbf{S}[\mathbf{u}^{\max} - \mathbf{u}(t)]$

$$\tilde{\mathbf{u}} = \mathbf{u}^{\max} - \mathbf{S}^{-1}[\frac{\epsilon \mathbf{w}(t)}{\gamma\lambda_3(t)} - \mathbf{a}]$$

$$u_k^*(t) = \begin{cases} u_k^{\min} : & \epsilon w_k(t) > \gamma\lambda_3(t)(a_k + s_k) \\ \tilde{u}_k : & \gamma\lambda_3(t) a_k < \epsilon w_k(t) < \gamma\lambda_3(t)(a_k + s_k) \\ u_k^{\max} : & \epsilon w_k(t) < \gamma\lambda_3(t) a_k \end{cases}$$

Resources and further reading on pandemic modeling and forecasting

Resources

1. The theoretical details of this research: `https://arxiv.org/abs/2003.11371`
2. The source codes of this research: `https://github.com/rsameni/EpidemicModeling.git`
3. COVID-19 real-time data: `https://www.worldometers.info/coronavirus/`
4. Johns Hopkins University's CSSE Git repository: `https://github.com/CSSEGISandData/COVID-19`

Further reading

1. Epidemic models: [Brauer et al., 2012]
2. Biological systems modeling: [Haefner, 2005, de Vries et al., 2006]
3. Stochastic aspects of mathematical epidemiology: [Britton, 2010, Pellis et al., 2012, Brauer et al., 2012, Miller, 2019]
4. Optimal estimation and Kalman filtering: [Grewal and Andrews, 2001]
5. Linear systems theory: [Kailath, 1980]
6. An interesting video on agent-based methods (the micro-modeling approach): `https://youtu.be/gxAaO2rsdIs`

Outline

Outline

The Kalman filter revisited

For the discrete-time stochastic dynamic system:

State equation	$\boldsymbol{s}_{n+1} = \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{B}_n \boldsymbol{w}_n$
Observation equation	$\boldsymbol{x}_n = \boldsymbol{C}_n \boldsymbol{s}_n + \boldsymbol{D}_n \boldsymbol{v}_n$

where $\boldsymbol{w}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{Q}_n)$, $\boldsymbol{v}_n \sim \mathcal{N}(\mathbf{0}, \boldsymbol{R}_n)$ and $\hat{\boldsymbol{s}}_0 = \mathbb{E}\{\boldsymbol{s}_0\}$, the Kalman filter is a set of recursive equations for estimating the state vector:

1. **Time propagation:**

Prior state estimate	$\hat{\boldsymbol{s}}^-_{n+1} = \boldsymbol{A}_n \hat{\boldsymbol{s}}_n$
Prior state covariance matrix	$\boldsymbol{P}^-_{n+1} = \boldsymbol{A}_n \boldsymbol{P}_n \boldsymbol{A}_n^T + \boldsymbol{B}_n \boldsymbol{Q}_n \boldsymbol{B}_n^T$
Kalman gain	$\boldsymbol{K}_n = \boldsymbol{P}^-_n \boldsymbol{C}_n^T (\boldsymbol{C}_n \boldsymbol{P}^-_n \boldsymbol{C}_n^T + \boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T)^{-1}$
Innovation process	$\boldsymbol{i}_n = \boldsymbol{x}_n - \boldsymbol{C}_n \hat{\boldsymbol{s}}^-_n$

2. **Measurement propagation:**

Posterior state estimate	$\hat{\boldsymbol{s}}_n = \hat{\boldsymbol{s}}^-_n + \boldsymbol{K}_n(\boldsymbol{x}_n - \boldsymbol{C}_n \hat{\boldsymbol{s}}^-_n) = \hat{\boldsymbol{s}}^-_n + \boldsymbol{K}_n \boldsymbol{i}_n$
Posterior state covariance matrix	$\boldsymbol{P}_n = \boldsymbol{P}^-_n - \boldsymbol{K}_n \boldsymbol{C}_n \boldsymbol{P}^-_n$

The Kalman filter revisited

(continued)

Corner cases

- For stationary models ($\boldsymbol{A}_n = \boldsymbol{A}$, $\boldsymbol{B}_n = \boldsymbol{B}$, $\boldsymbol{C}_n = \boldsymbol{C}$, $\boldsymbol{D}_n = \boldsymbol{D}$, $\boldsymbol{Q}_n = \boldsymbol{Q}$, $\boldsymbol{R}_n = \boldsymbol{R}$), the KF converges to the causal **Wiener filter** in steady-state.
- With extremely noisy measurements (or extremely good dynamic models) it becomes **a recursive solver of the state-space model** from a random initial condition. A Monte Carlo simulation over different initial conditions gives a picture of the solution space.
- Without priors (very inaccurate models, or very good measurements), the Kalman filter gives the **maximum likelihood**/**least squares** estimates of the states[a].

[a] ML vs LS interpretation of the KF solution depends on whether or not the measurement noise is presumed to be gaussian. With Gaussian noise ML and LS are identical.

In control theory, the KF has other interpretations in terms of *an optimal state observer*, and *state controller*, which are beyond the scope of this lecture.

Interpretation of Time propagation

$\hat{\boldsymbol{s}}_{n+1}^{-} = \boldsymbol{A}_n \hat{\boldsymbol{s}}_n$

Before receiving the nth observation, propagate the latest state estimate by using the state-transition matrix.

Question: Can we extend this idea? **Yes!** If there are missing measurements, or if we want to perform a k-step ahead prediction:

$$\hat{\boldsymbol{s}}_{n+k}^{-} = \underbrace{\boldsymbol{A}_{n+k-1} \cdots \boldsymbol{A}_{n+1} \boldsymbol{A}_n}_{\phi_{n \to n+k}} \hat{\boldsymbol{s}}_n$$

Non-zero mean process noise

The average process noise adds an offset to the state estimate: $\hat{\boldsymbol{s}}_{n+1}^{-} = \boldsymbol{A}_n \hat{\boldsymbol{s}}_n + \boldsymbol{B}_n \mathbb{E}\{\boldsymbol{w}_n\} = \boldsymbol{A}_n \hat{\boldsymbol{s}}_n + \boldsymbol{B}_n \bar{\boldsymbol{w}}_n$

Interpretation of Time propagation

(continued)

$$\boldsymbol{P}_{n+1}^{-} = \boldsymbol{A}_n \boldsymbol{P}_n \boldsymbol{A}_n^T + \boldsymbol{B}_n \boldsymbol{Q}_n \boldsymbol{B}_n^T$$

This is directly obtained from the system's dynamics ($\boldsymbol{s}_{n+1} = \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{B}_n \boldsymbol{w}_n$) and the uncorrelatedness of the process noise and the current state (*before the new observation arrives*).

The innovation process

Can we predict the next observation in advance (before its arrival/measurement)?

How about the following? $\hat{\boldsymbol{x}}_n = \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^-$

Question: What if the observation noise is non-zero mean?

The average process noise adds an offset to the state estimate: $\hat{\boldsymbol{x}}_n = \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^- + \boldsymbol{D}_n \mathbb{E}\{\boldsymbol{v}_n\} = \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^- + \bar{\boldsymbol{v}}_n$

The innovation process

- Is the error between the observations and and its estimate: $\boldsymbol{i}_n = \boldsymbol{x}_n - \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^-$
- It is the only innovative (unpredictable) part of the observation, i.e., the remaining part of the observation ($\hat{\boldsymbol{x}}_n = \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^-$) was already predictable before observation
- The innovations set $(\boldsymbol{i}_0, \boldsymbol{i}_1, \ldots, \boldsymbol{i}_n)$ is equivalent with the observations $(\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)$

Measurement update

Question: Propose a linear predictor for the states by combining the prior (before observation) and posteriors (after observations).

General form of linear estimators

$$\begin{aligned}\hat{\boldsymbol{s}}_n &= \hat{\boldsymbol{s}}_n^- + \boldsymbol{K}_n \boldsymbol{i}_n \\ &= \hat{\boldsymbol{s}}_n^- + \boldsymbol{K}_n(\boldsymbol{x}_n - \boldsymbol{C}_n \hat{\boldsymbol{s}}_n^-) \\ &= (\boldsymbol{I} - \boldsymbol{K}_n \boldsymbol{C}_n)\hat{\boldsymbol{s}}_n^- + \boldsymbol{K}_n \boldsymbol{x}_n\end{aligned}$$

where $\boldsymbol{K}_n$ is an arbitrary gain matrix (*a transformation from the observation space to the state space*).

The Kalman gain

Rudolf E. Kálmán's elegant contribution was the **optimal** gain, for which the general linear estimator becomes the **"minimum mean square error"** estimator:

$$\boldsymbol{K}_n = \boldsymbol{P}_n^- \boldsymbol{C}_n^T (\boldsymbol{C}_n \boldsymbol{P}_n^- \boldsymbol{C}_n^T + \boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T)^{-1}$$

Prior vs posterior covariance matrices

$$\boldsymbol{P}_n = \boldsymbol{P}_n^- - \boldsymbol{K}_n\boldsymbol{C}_n\boldsymbol{P}_n^- = \boldsymbol{P}_n^- - \boldsymbol{P}_n^-\boldsymbol{C}_n^T(\boldsymbol{C}_n\boldsymbol{P}_n^-\boldsymbol{C}_n^T + \boldsymbol{D}_n\boldsymbol{R}_n\boldsymbol{D}_n^T)^{-1}\boldsymbol{C}_n\boldsymbol{P}_n^-$$

- As covariance matrices, $\boldsymbol{P}_n^-$ and $\boldsymbol{P}_n$ are symmetric semi-positive definite.
- We can show that regardless of the KF performance $\boldsymbol{P}_n \preccurlyeq \boldsymbol{P}_n^-$, where $\boldsymbol{A} \preccurlyeq \boldsymbol{B}$ denotes $\boldsymbol{B} - \boldsymbol{A}$ being non-negative definite.
- This implies that using the observations always reduces, or at least preserves, the estimation error. In the vector case, the area of the prior/posterior covariance matrix ellipsoids can be compared.

Outline

Kalman filter parameter selection

- In addition to the dynamic model and its parameters (which are governed by the physics/biology of the problem of interest), the Kalman filter also requires the selection of the initial state covariance matrix $\boldsymbol{P}_0$, the process noise covariance matrix $\boldsymbol{Q}_n$ and the observation noise covariance matrix $\boldsymbol{R}_n$.
- The correct selection of $\boldsymbol{P}_0$, $\boldsymbol{Q}_n$ and $\boldsymbol{R}_n$ is essential for the performance of the Kalman filter.
- Whenever the precise selection of these parameters is not possible (due to lack of knowledge), as a rule of thumb, it is better to overestimate them by selecting some upper bounds $\tilde{\boldsymbol{P}}_0$, $\tilde{\boldsymbol{Q}}_n$ and $\tilde{\boldsymbol{R}}_n$, such that: $\tilde{\boldsymbol{P}}_0 \succcurlyeq \boldsymbol{P}_0$, $\tilde{\boldsymbol{Q}}_n \succcurlyeq \boldsymbol{Q}_n$, $\tilde{\boldsymbol{R}}_n \succcurlyeq \boldsymbol{R}_n$[1].
- It can be shown that by overestimating these parameters, the covariance matrices reported by the KF is also an overestimate of the true values (i.e., the filter is practically working better than what it reports).
- Note that for a stable system, the effect of the initial state estimate $\hat{\boldsymbol{s}}_0$ vanishes over time.

[1] $\boldsymbol{A} \succcurlyeq \boldsymbol{B}$ denotes $\boldsymbol{A} - \boldsymbol{B}$ being a semi-positive definite matrix.

Whiteness of the innovation process

- For a well-designed KF with well-selected parameters, the innovation process is zero-mean and spectrally white
- **Interpretation:** when the parameters of a KF are correctly selected, the filter uses all the "predictable" parts of the observations, leaving white noise residuals (the innovations)
- Testing the spectral whiteness of the innovations is a necessary sanity check for a "healthy Kalman filter"
- Non-white innovations indicate mis-selection of KF parameters, or deviation of the system parameters (very useful for fault detection, nonstationary event detection, etc.)

A test of innovations process spectral whiteness

- The *data-driven* cross-correlation matrix estimate of the innovations process:

$$\boldsymbol{\rho}_n(\tau) = \frac{1}{2w}\sum_{k=0}^{w-1}[\boldsymbol{i}_{n-k}\boldsymbol{i}_{n-k-\tau}^T + \boldsymbol{i}_{n-k-\tau}\boldsymbol{i}_{n-k}^T] \quad (**)$$

  over a window of length w should be ideally zero for $\tau \neq 0$.
- Parametric and non-parametric tests can be performed on $\boldsymbol{\rho}_n(\tau)$ to check the whiteness of the innovations.

KF parameter selection sanity check

- Beyond being zero-mean and spectrally white, the innovation process covariance matrix $\boldsymbol{\rho} = \mathbb{E}\{\boldsymbol{i}_n \boldsymbol{i}_n^T\}$ should also be *ideally* equal to $\boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T$.

  Question: Why?
- If $\boldsymbol{D}_n \boldsymbol{R}_n \boldsymbol{D}_n^T$ and the data-driven innovations covariance matrix $\boldsymbol{\rho}_n(\tau)$ in $(**)$ are far from being equal, it is an indication of mis-selected KF parameters or nonstationary events within the system/observations.
- This property can be used to design an index for KF sanity checking or nonstationarity detection $\rightarrow$ *has applications in EEG and ECG even detection.*

A test of innovation process covariance correctness

- The cross-correlation matrix estimate of the innovations process:

$$\boldsymbol{r}_n = \frac{1}{w} \sum_{k=n-w+1}^{n} \boldsymbol{\rho}_k(0)(\boldsymbol{C}_k \boldsymbol{P}_k^- \boldsymbol{C}_k^T + \boldsymbol{D}_k \boldsymbol{R}_k \boldsymbol{D}_k^T)^{-1}$$

  should be close to identity.

  Question: Why?
- Parametric and non-parametric tests can be performed on $\boldsymbol{r}_n$ to check this property.
- Depending on the *expected source of error*, $\boldsymbol{Q}_n$ and/or $\boldsymbol{R}_n$ can be modified to satisfy the property.

Stability

- Stability/Instability is a property of the underlying dynamic model of the system.
- We can show that for the dynamic model $\boldsymbol{s}_{n+1} = \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{B}_n \boldsymbol{w}_n$, the system is stable *iff* $|\lambda(z\boldsymbol{I} - \boldsymbol{A}_n)| < 1$, where $\lambda(\cdot)$ denotes the eigenvalues.
- Under some general conditions on the dynamic model parameters, the Kalman filter can converge and track the state vectors, even in some case, for unstable dynamic systems. In other words, a system's dynamics may be unstable, while the Kalman filter that tracks its states is stable (cf. the Further Reading textbooks).

Observability

- **Question:** Is state estimation from noisy measurements always possible?
- It depends on the system's dynamics. Intuitively, the observations should contain some information about the states, to enable estimation (the observations should have "seen the states").
- More formally:

Observability

The state-space model

$$\begin{aligned} \boldsymbol{s}_{n+1} &= \boldsymbol{A}_n \boldsymbol{s}_n + \boldsymbol{B}_n \boldsymbol{w}_n \\ \boldsymbol{x}_n &= \boldsymbol{C}_n \boldsymbol{s}_n + \boldsymbol{D}_n \boldsymbol{v}_n \end{aligned} \qquad (*)$$

is **observable** from its outputs, if the state variables can be estimated from the observations in finite time [Kailath, 1980].

Observability rank-test

The dynamic system $(*)$ is observable if the following matrix (known as the **observability matrix**) is rank n (the number of state variables):

$$\mathcal{O}_k = \begin{bmatrix} \boldsymbol{C}_k \\ \boldsymbol{C}_k \boldsymbol{A}_k \\ \dots \\ \boldsymbol{C}_k \boldsymbol{A}_k^{n-1} \end{bmatrix}$$

Offline calculations of estimation quality

- In the classical Kalman filter equations, the Kalman gain and the prior/posterior state covariance matrices are independent of the observations.
- In other words, assuming that the model parameters are known *a priori*, we can estimate the Kalman filter and its quality, before having any real measurements (provided that the measurement noise variance is known in advance).
- Depending on the application, using the above property, the Kalman gain and the error covariance matrices can be calculated offline and stored in the memory. $\rightarrow$ Useful for limit resource, low-power, or edge computing applications.

Numerical issues

- Covariance matrices are inherently **symmetric** and **semi-positive definite**.
- In order to guarantee the preservation of this property during the KF covariance updates, alternative forms of the covariance matrix update can be used.

Joseph stabilized posterior covariance matrix update

$$\begin{aligned} \boldsymbol{P}_n &= \boldsymbol{P}_n^- - \boldsymbol{K}_n\boldsymbol{C}_n\boldsymbol{P}_n^- && \textbf{Ordinary form} \\ &= (\boldsymbol{I} - \boldsymbol{K}_n\boldsymbol{C}_n)\boldsymbol{P}_n^-(\boldsymbol{I} - \boldsymbol{K}_n\boldsymbol{C}_n)^T + \boldsymbol{K}_n\boldsymbol{R}_n\boldsymbol{K}_n^T && \textbf{Joseph stabilized form} \end{aligned}$$

Symmetry

After the calculation of the prior and posterior updates, the following steps can be added (which cancel out any asymmetric terms added due to numerical round-off errors):

$$\boldsymbol{P}_n^- \leftarrow \frac{1}{2}(\boldsymbol{P}_n^- + \boldsymbol{P}_n^{-T}), \quad \boldsymbol{P}_n \leftarrow \frac{1}{2}(\boldsymbol{P}_n + \boldsymbol{P}_n^T)$$

Continuous-time Kalman filter

Consider the continuous-time stochastic dynamic system:

$$\begin{aligned} \text{State equation} \quad \dot{\boldsymbol{s}}(t) &= \boldsymbol{F}(t)\boldsymbol{s}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t) \\ \text{Observation equation} \quad \boldsymbol{x}(t) &= \boldsymbol{M}(t)\boldsymbol{s}(t) + \boldsymbol{v}(t) \end{aligned}$$

where $\mathbb{E}\{\boldsymbol{w}(t)\} = 0$, $\mathbb{E}\{\boldsymbol{w}(t)\boldsymbol{w}(t-\tau)\} = \boldsymbol{Q}(t)\delta(t-\tau)$, $\mathbb{E}\{\boldsymbol{v}(t)\} = 0$, $\mathbb{E}\{\boldsymbol{v}(t)\boldsymbol{v}(t-\tau)\} = \boldsymbol{R}(t)\delta(t-\tau)$ and $\hat{\boldsymbol{s}}(0) = \mathbb{E}\{\boldsymbol{s}(0)\}$. Denoting $\hat{\boldsymbol{s}}(t)$ as the estimate of $\boldsymbol{s}(t)$ and its covariance $\hat{\boldsymbol{P}}(t)$ given the observations $\{\boldsymbol{x}(\tau)|0 \leq \tau \leq t\}$, the continuous-time Kalman filter equations are:

$$\begin{aligned} \text{Kalman gain} \quad \boldsymbol{K}(t) &= \hat{\boldsymbol{P}}(t)\boldsymbol{M}(t)^T\boldsymbol{R}(t)^{-1} \\ \text{Innovation process} \quad \boldsymbol{i}(t) &= \boldsymbol{x}(t) - \boldsymbol{M}(t)\hat{\boldsymbol{s}}(t) \\ \text{State estimate differential equation} \quad \frac{d}{dt}\hat{\boldsymbol{s}}(t) &= \boldsymbol{F}(t)\hat{\boldsymbol{s}}(t) + \boldsymbol{K}(t)\boldsymbol{i}(t) \\ \text{Riccati differential equation} \quad \frac{d}{dt}\hat{\boldsymbol{P}}(t) &= \boldsymbol{F}(t)\hat{\boldsymbol{P}}(t) + \hat{\boldsymbol{P}}(t)\boldsymbol{F}(t)^T + \boldsymbol{G}(t)\boldsymbol{Q}(t)\boldsymbol{G}(t)^T - \boldsymbol{K}(t)\boldsymbol{M}(t)\hat{\boldsymbol{P}}(t) \end{aligned}$$

The Riccati equation can be solved numerically, by MATLAB and other optimization software.

Continuous-time state space model with discrete observations

- A very practical extension of the Kalman filter (especially for physical and biological systems) is for systems governed by continuous-time dynamics, observed through discrete-time observations:

$$\begin{aligned} \boxed{\text{State equation}} \quad \dot{\boldsymbol{s}}(t) &= \boldsymbol{F}(t)\boldsymbol{s}(t) + \boldsymbol{G}(t)\boldsymbol{w}(t) \\ \boxed{\text{Observation equation}} \quad \boldsymbol{x}_n &= \boldsymbol{C}_n\boldsymbol{s}(t_n) + \boldsymbol{v}(t_n) \end{aligned} \tag{1}$$

  where t_n are the uniform or non-uniform sample times.
- One approach is to discretize the state equations and then use a discrete-time KF.
- A better approach is to use the combination of continuous-time dynamics and discrete-time observations as they are.

Continuous-time state space model with discrete observations
(continued)

- Integrating (1) between t_n and t_{n+1} ($\Delta_n = t_{n+1} - t_n$) we find:

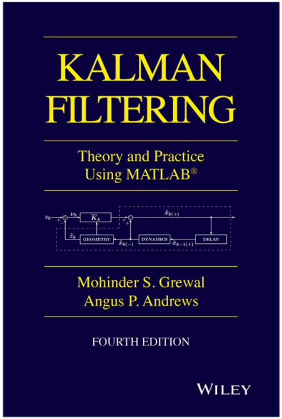
$$\boldsymbol{s}(t_{n+1}) = \phi(t_n, t_{n+1})\boldsymbol{s}(t_n) + \underbrace{\int_{t_n}^{t_{n+1}} \phi(t_n, \tau)\boldsymbol{G}(\tau)\boldsymbol{w}(\tau)d\tau}_{\tilde{\boldsymbol{w}}_n}$$

  where $\phi(t, \tau)$ is the state transition matrix, which satisfies: $\phi(t_0, t_0) = \boldsymbol{I}$ and $\dot{\phi}(t, t_0) = \boldsymbol{F}(t)\phi(t, t_0)$. If $\boldsymbol{F}(\cdot)$ is rather constant from t_n to t_{n+1}, then $\phi(t_n, t_{n+1}) \approx \exp(\boldsymbol{F}(\Delta_n t_n)) \triangleq \boldsymbol{A}_n$. Where for small Δ_n, $\boldsymbol{A}_n \approx \boldsymbol{I} + \Delta_n \boldsymbol{F}(t_n)$. Moreover,
  $\tilde{\boldsymbol{Q}}_n = \mathbb{E}\{\tilde{\boldsymbol{w}}_n \tilde{\boldsymbol{w}}_n^T\} = \int_0^{\Delta_n} \exp(\zeta \boldsymbol{F}(\zeta))\boldsymbol{G}(\zeta)\boldsymbol{Q}(\zeta)\boldsymbol{G}(\zeta)^T \exp(\zeta \boldsymbol{F}(\zeta))^T d\zeta \approx \Delta_n \boldsymbol{G}(\zeta)\boldsymbol{Q}(\zeta)\boldsymbol{G}(\zeta)^T$.
- The linearized form of the dynamic equation now reads:

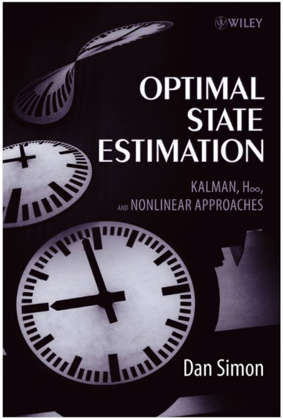
$$\begin{aligned} \boldsymbol{s}(t_{n+1}) &\approx \boldsymbol{A}_n \boldsymbol{s}(t_n) + \tilde{\boldsymbol{w}}_n \\ \boldsymbol{x}_n &= \boldsymbol{C}_n \boldsymbol{s}(t_n) + \boldsymbol{v}(t_n) \end{aligned}$$

- The above form can now be used in a discrete-time Kalman filter framework.

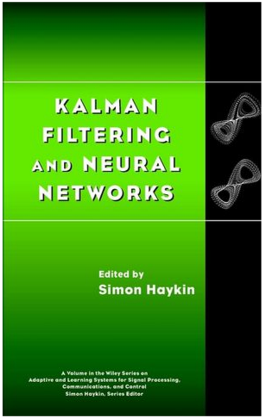
Further reading



M. S. Grewal and A. P. Andrews, Kalman Filtering: Theory and Practice Using Matlab, 4th ed. John Wiley & Sons, Inc., 2015.



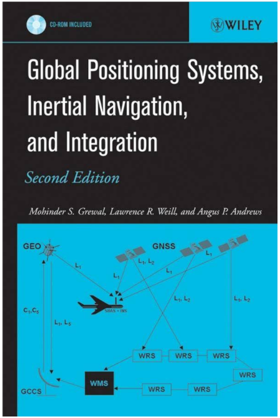
D. Simon, Optimal State Estimation: Kalman, H Infinity, and Nonlinear Approaches. John Wiley & Sons Inc., 2006.



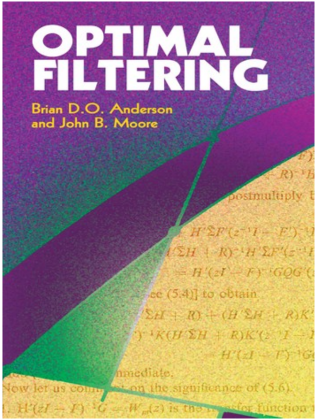
S. Haykin, Ed., Kalman Filtering and Neural Networks. John Wiley & Sons Inc., 2001.

Further reading
(continued)

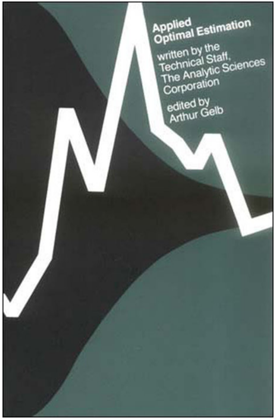
M. S. Grewal, L. Weill, and A. Andrews, Global Positioning Systems, Inertial Navigation, and Integration. Wiley, 2007
→*Chapter 8 for some nice Kalman filter engineering techniques*



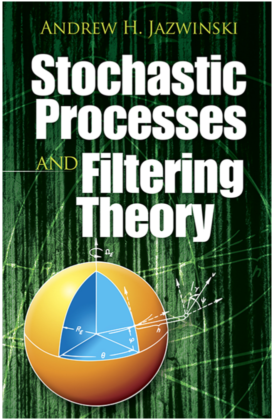
Oldies but goldies!



Brian D.O. Anderson and John B. Moore. Optimal filtering. Dover Publication Incorporation, 1979.



Arthur Gelb ed. Applied optimal estimation. MIT press, 1974.



Andrew H. Jazwinski. Stochastic processes and filtering theory. Dover Publication Incorporation, 1970.

Outline

Kalman filter lab session requirements

Requirements

- Clone the Open-Source Electrophysiological Toolbox (OSET): `https://gitlab.com/rsameni/OSET.git` (or its image repository on GitHub: `https://github.com/alphanumericslab/OSET.git`)

Kalman filter lab session

Reference to OSET MATLAB scripts studied in the lab session

`testKalmanARFilter.m`	ECG Baseline wander removal using a first-order AR model for the baseline
`testKFNotch.m`	Powerline noise cancellation using KF
`testKFNotchStability.m`	Visualize the KF notch filter frequency responses in steady-state
`testECGKalmanFilter.m`	ECG Kalman filtering and smoothing
`testFECGKalmanFilter1.m`	Maternal ECG cancellation in DATASY dataset
`testFECGKalmanFilter2.m`	Post-process fetal ECG by KF in DAISY dataset
`testFECGExtractionByEKF.m`	Maternal ECG cancellation

References I

L. D. Avendaño-Valencia, L. E. Avendaño, J. Ferrero, and Castellanos-Domínguez. Improvement of an extended Kalman filter power line interference suppressor for ECG signals. In *2007 Computers in Cardiology*, pages 553 –556. IEEE, Sept. 2007. doi: 10.1109/cic.2007.4745545.

M. J. Barton, P. A. Robinson, S. Kumar, A. Galka, H. F. Durrant-Whyte, J. Guivant, and T. Ozaki. Evaluating the Performance of Kalman-Filter-Based EEG Source Localization. *IEEE Transactions on Biomedical Engineering*, 56(1): 122–136, 2009. doi: 10.1109/TBME.2008.2006022.

J. Behar. *Extraction of clinical information from the non-invasive fetal electrocardiogram*. PhD thesis, Balliol College, University of Oxford, UK, September 2014. URL https://arxiv.org/abs/1606.01093.

F. Brauer, C. Castillo-Chavez, and C. Castillo-Chavez. *Mathematical models in population biology and epidemiology*, volume 2. Springer, 2012.

T. Britton. Stochastic epidemic models: a survey. *Mathematical biosciences*, 225(1):24–35, 2010.

B. De Moor. *Database for the Identification of Systems (DaISy)*, 1997. URL http://homes.esat.kuleuven.be/~smc/daisy/.

G. de Vries, T. Hillen, M. Lewis, B. Schõnfisch, and J. Muller. *A Course in Mathematical Biology: Quantitative Modeling with Mathematical and Computational Methods*. Monographs on Mathematical Modeling and Computation. Society for Industrial and Applied Mathematics, 2006. ISBN 9780898716122.

J. W. Gordon and J. O. Smith. A sine generation algorithm for VLSI applications. In *Proc. 1985 Int. Computer Music Conf., Vancouver*. Computer Music Association, Computer Music Association, 1985.

References II

M. S. Grewal and A. P. Andrews. *Kalman Filtering: Theory and Practice Using Matlab*. John Wiley & Sons, Inc., 2nd edition, 2001.

J. Haefner. *Modeling Biological Systems: Principles and Applications*. Springer, 2005. ISBN 9780387250113.

T. Kailath. *Linear Systems*. Prentice Hall, 1980.

E. A. Kheirati Roonizi. *Adaptive model-based cardiac signals analysis and feature extraction*. PhD thesis, School of Computer Science, University of Milan, Italy, 2017. URL https://air.unimi.it/retrieve/handle/2434/468106/764783/phd_unimi_R10774.pdf.

N. Kheram. Modeling and Denoising of Phonocardiogram Signals. Master's thesis, Biomedical Engineering, School of Electrical & Computer Engineering, Shiraz University, 9 2019. Supervised by: Dr. Reza Sameni.

Q. Li, R. G. Mark, and G. D. Clifford. Robust heart rate estimation from multiple asynchronous noisy sources using signal quality indices and a Kalman filter. *Physiological measurement*, 29(1):15, 2007.

P. E. McSharry, G. D. Clifford, L. Tarassenko, and L. A. Smith. A Dynamic Model for Generating Synthetic Electrocardiogram Signals. *Biomedical Engineering, IEEE Transactions on*, 50:289–294, mar 2003.

J. C. Miller. Distribution of outbreak sizes for SIR disease in finite populations. *arXiv preprint arXiv:1907.05138*, 2019.

V. Oikonomou, A. Tzallas, and D. Fotiadis. A Kalman filter based methodology for EEG spike enhancement. *Computer Methods and Programs in Biomedicine*, 85(2):101–108, 2007. ISSN 0169-2607. doi: https://doi.org/10.1016/j.cmpb.2006.10.003.

References III

A. H. Omidvarnia, M. Mesbah, M. S. Khlif, J. M. O'Toole, P. B. Colditz, and B. Boashash. Kalman filter-based time-varying cortical connectivity analysis of newborn EEG. In *2011 Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, pages 1423–1426, 2011. doi: 10.1109/IEMBS.2011.6090335.

L. Patomaki, J. Kaipio, and P. Karjalainen. Tracking of nonstationary EEG with the roots of ARMA models. In *Proceedings of 17th International Conference of the Engineering in Medicine and Biology Society*, volume 2, pages 877–878 vol.2, 1995. doi: 10.1109/IEMBS.1995.579249.

L. Pellis, F. Ball, and P. Trapman. Reproduction numbers for epidemic models with households and other social structures. I. Definition and calculation of R0. *Mathematical biosciences*, 235(1):85–97, 2012.

M. Rahbar Alam and R. Sameni. Automatic wake-sleep stages classification using electroencephalogram instantaneous frequency and envelope tracking. *bioRxiv*, 2020. URL http://dx.doi.org/10.1101/2020.05.13.092841.

R. Sameni. *Extraction of Fetal Cardiac Signals from an Array of Maternal Abdominal Recordings*. PhD thesis, Sharif University of Technology – Institut National Polytechnique de Grenoble, July 2008. URL http://www.sameni.info/Publications/Thesis/PhDThesis.pdf.

R. Sameni. A Linear Kalman Notch Filter for Power-Line Interference Cancellation. In *Proceedings of the 16th CSI International Symposium on Artificial Intelligence and Signal Processing (AISP)*, pages 604–610, Shiraz, Iran, 2-3 May 2012 2012. URL https://doi.org/10.1109/AISP.2012.6313817.

R. Sameni. Mathematical modeling of epidemic diseases; a case study of the COVID-19 coronavirus. *arXiv preprint*, 2020. URL https://arxiv.org/abs/2003.11371.

References IV

R. Sameni. *Noninvasive Fetal Electrocardiography: Models, Technologies, and Algorithms*, pages 99–146. Springer International Publishing, Cham, 2021a. ISBN 978-3-030-54403-4. URL https://doi.org/10.1007/978-3-030-54403-4_5.

R. Sameni. Model-based prediction and optimal control of pandemics by nonpharmaceutical interventions. *arXiv preprint*, 2021b. URL https://arxiv.org/abs/2102.06609.

R. Sameni, M. B. Shamsollahi, and C. Jutten. Filtering Electrocardiogram Signals Using the Extended Kalman Filter. In *Proceedings of the 27th Annual International Conference of the IEEE Engineering in Medicine and Biology Society (EMBS)*, pages 5639–5642, Shanghai, China, September 1-4 2005. URL https://doi.org/10.1109/IEMBS.2005.1615765.

R. Sameni, M. Shamsollahi, and C. Jutten. Multi-Channel Electrocardiogram Denoising Using a Bayesian Filtering Framework. In *Proc. of the 33rd Annual International Conference on Computers in Cardiology*, pages 185–188, Valencia, Spain, September 17-20 2006. URL http://cinc.mit.edu/archives/2006/.

R. Sameni, M. B. Shamsollahi, C. Jutten, and G. D. Clifford. A nonlinear bayesian filtering framework for ECG denoising. *Biomedical Engineering, IEEE Transactions on*, 54(12):2172–2185, December 2007. URL https://doi.org/10.1109/TBME.2007.897817.

R. Sameni, C. Jutten, and M. B. Shamsollahi. Multichannel Electrocardiogram Decomposition using Periodic Component Analysis. *Biomedical Engineering, IEEE Transactions on*, 55(8):1935–1940, Aug 2008a. URL https://doi.org/10.1109/TBME.2008.919714.

References V

R. Sameni, M. B. Shamsollahi, and C. Jutten. Model-based Bayesian filtering of cardiac contaminants from biomedical recordings. *Physiological Measurement*, 29(5):595–613, May 2008b. URL https://doi.org/10.1088/0967-3334/29/5/006.

R. Sameni, C. Jutten, M. Shamsollahi, and G. Clifford. Extraction of Fetal Cardiac Signals, June 2010. Licensed to MindChild Medical Inc.

B. L. Scala and R. Bitmead. Design of an extended Kalman filter frequency tracker. *IEEE Transactions on Signal Processing*, 44(3):739–742, Mar. 1996. doi: 10.1109/78.489052.

G. J. J. Warmerdam, R. Vullings, L. Schmitt, J. O. E. H. V. Laar, and J. W. M. Bergmans. A Fixed-Lag Kalman Smoother to Filter Power Line Interference in Electrocardiogram Recordings. *IEEE Transactions on Biomedical Engineering*, 64(8):1852–1861, Aug. 2017. doi: 10.1109/tbme.2016.2626519.